A strict minimax inequality criterion and some of its consequences

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Abstract: In this paper, we point out a very flexible scheme within which a strict minimax inequality occurs. We then show the fruitfulness of this approach presenting a series of various consequences. Here is one of them:

Let Y be a finite-dimensional real Hilbert space, $J : Y \to \mathbf{R}$ a C^1 function with locally Lipschitzian derivative, and $\varphi : Y \to [0, +\infty[$ a C^1 convex function with locally Lipschitzian derivative at 0 and $\varphi^{-1}(0) = \{0\}$.

Then, for each $x_0 \in Y$ for wich $J'(x_0) \neq 0$, there exists $\delta > 0$ such that, for each $r \in]0, \delta[$, the restriction of J to $B(x_0, r)$ has a unique global minimum u_r which satisfies

$$J(u_r) \leq J(x) - \varphi(x - u_r)$$

for all $x \in B(x_0, r)$, where $B(x_0, r) = \{x \in Y : \|x - x_0\| \leq r\}$.

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1. Introduction

Let X, Λ be two non-empty sets and $f : X \times \Lambda \to \mathbf{R}$ a given function. Clearly, we have

$$\sup_{\lambda \in \Lambda} \inf_{x \in X} f(x, \lambda) \leq \inf_{x \in X} \sup_{\lambda \in \Lambda} f(x, \lambda) .$$

The classical minimax theory deals with minimax theorems, that is to say with results which ensure the validity of the equality

$$\sup_{\lambda \in \Lambda} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in \Lambda} f(x, \lambda) .$$

In this paper, to the contrary, we are interested in the strict inequality

$$\sup_{\lambda \in \Lambda} \inf_{x \in X} f(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in \Lambda} f(x, \lambda) . \tag{1}$$

While the importance of minimax theorems in various fields is well established since long time, the interest in the study of the strict minimax inequality is much more recent, being been originated in [12] and [5]. Actually, in [12], the following result was obtained:

THEOREM A. - *Let X be a compact metric space, $\Lambda \subset \mathbf{R}$ a compact interval and $f : X \times \Lambda \to \mathbf{R}$ a function which is lower semicontinuous in X and continuous and concave in Λ. Assume also that (1) holds.*

Then, there exists an open interval $A \subset \Lambda$ such that, for each $\lambda \in A$, the function $f(\cdot, \lambda)$ has a local, not global minimum.

In turn, in [5], Theorem A was used to get the following

THEOREM B. - *Let X be a separable and reflexive real Banach space, $\Lambda \subseteq \mathbf{R}$ an interval, and $f : X \times \Lambda \to \mathbf{R}$ a continuous function which is concave in Λ, and sequentially weakly lower semicontinuous, coercive, and satisfying the Palais-Smale condition in X. Assume also that (1) holds.*

Then, there exist an open interval $A \subseteq \Lambda$ and a positive real number ρ, such that, for each $\lambda \in \Lambda$, the equation

$$f'_x(x, \lambda) = 0$$

has at least three solutions in X whose norms are less than ρ.

When $f(x, \cdot)$ is affine, Theorem B is one of the most often applied multiplicty results in dealing with problems of variational nature. In this connection, we refer to [8] for a comprehensive account of the relevant literature.

A further very recent motivation for the study of (1) comes from [9] and [3]. In particular, in [3], S. J. N. Mosconi proved the following

THEOREM C. - *Let X be a topological space, $\Lambda \subseteq \mathbf{R}^n$ a convex set with non-empty interior, $f : X \times \Lambda \to \mathbf{R}$ a function which is lower semicontinuous and inf-compact in X, and upper semicontinuous and concave in Λ. Assume also that (1) holds.*

Then, there exists $\lambda^* \in \mathrm{int}(\Lambda)$ such that the function $f(\cdot, \lambda^*)$ has at least two global minima.

More precisely, Theorem C extends to the case $n > 1$ a result previously established in [9] (for $n = 1$) under the less restrictive assumption of quasi-concavity for $f(x, \cdot)$ (see also [14]). In turn, in Section 3, we will extend Theorem C to the case where Λ is a convex set in an arbitrary Hausdorff topological vector space (Theorem 3.2).

It is worth noticing that Theorem A, to the contrary, does not admit such an extension. Indeed, take $X = \{(t, s) \in \mathbf{R}^2 : t^2 + s^2 = 1\}$, $\Lambda = \{(\mu, \nu) \in \mathbf{R}^2 : \mu^2 + \nu^2 \leq 1\}$ and

$$f(t, s, \mu, \nu) = \mu t + \nu s$$

for all $(t, s, \mu, \nu) \in X \times \Lambda$. Of course, we have

$$\sup_{(\mu,\nu)\in\Lambda} \inf_{(t,s)\in X} f(t, s, \mu, \nu) = 0 < 1 = \inf_{(t,s)\in X} \sup_{(\mu,\nu)\in\Lambda} f(t, s, \mu, \nu) .$$

Moreover, the function f is continuous in $X \times \Lambda$ and concave in Λ. Nevertheless, for each $(\mu, \nu) \in \Lambda$, the function $f(\cdot, \cdot, \mu, \nu)$ has no local, not global minima in X.

The aim of this paper is to establish Theorem 3.1 below (whose formulation has been inspired by Proposition 1 of [10]) which provides a very flexible scheme within which (1) is obtained.

Just to stress the great flexibility of Theorem 3.1, we now state six of its consequences, of a quite different nature, whose proofs will be given in Section 3.

First, we fix a few relevant definitions.

A non-empty set C in a normed space S is said to be uniquely remotal with respect to a set $D \subseteq S$ if, for each $y \in D$, there exists a unique $x \in C$ such that

$$\|x - y\| = \sup_{u \in C} \|u - y\| \ .$$

The main problem in theory of such sets is to know if they are singletons.

If E, F are two real vector spaces and D is a convex subset of E, we say that an operator $\Phi : D \to F$ is affine if

$$\Phi(\lambda x + (1 - \lambda)y) = \lambda \Phi(x) + (1 - \lambda)\Phi(y)$$

for all $x, y \in D$, $\lambda \in [0, 1]$.

Let $(T, \mathcal{F}, \mu)$ be a measure space, E a real Banach space and $p \geq 1$.

As usual, $L^p(T, E)$ denotes the space of all (equivalence classes of) strongly μ-measurable functions $u : T \to E$ such that $\int_T \| u(t) \|^p \, d\mu < +\infty$, equipped with the norm

$$\| u \|_{L^p(T,E)} = \left(\int_T \| u(t) \|^p \, d\mu \right)^{\frac{1}{p}} \ .$$

A set $D \subseteq L^p(T, E)$ is said to be decomposable if, for every $u, v \in D$ and every $A \in \mathcal{F}$, the function

$$t \to \chi_A(t)u(t) + (1 - \chi_A(t))v(t)$$

belongs to D, where χ_A denotes the characteristic function of A.

Here is the series of statements announced above.

THEOREM 1.1. - *Let Y be a real normed space and let $X \subseteq Y$ be a non-empty compact uniquely remotal set with respect to $\mathrm{conv}(X)$.*
Then, X is a singleton.

THEOREM 1.2. - *Let X be a finite-dimensional real Hilbert space and $J : X \to \mathbf{R}$ a C^1 function. Set*

$$\eta = \liminf_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2}$$

and

$$\theta = \inf \left\{ \frac{J(x) - J(u)}{\|x - u\|^2} : (u, x) \in M_J \times X \text{ with } x \neq u \right\}$$

where M_J denotes the set of all global minima of J. Assume that

$$\theta < \eta \ .$$

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Then, for each $\mu \in]2\theta, 2\eta[$, there exists $y_\mu \in X$ such that the equation

$$J'(x) - \mu x = y_\mu$$

has at least three solutions.

THEOREM 1.3. - *Let Y be a finite-dimensional real Hilbert space, $J : Y \to \mathbf{R}$ a C^1 function with locally Lipschitzian derivative, and $\varphi : Y \to [0, +\infty[$ a C^1 convex function with locally lipschitzian derivative at 0 and $\varphi^{-1}(0) = \{0\}$.*

Then, for each $x_0 \in Y$ for wich $J'(x_0) \neq 0$, there exists $\delta > 0$ such that, for each $r \in]0, \delta[$, the restriction of J to $B(x_0, r)$ has a unique global minimum u_r which satisfies

$$J(u_r) \leq J(x) - \varphi(x - u_r)$$

for all $x \in B(x_0, r)$, where

$$B(x_0, r) = \{x \in Y : \|x - x_0\| \leq r\} .$$

THEOREM 1.4. - *Let X be a convex subset of a real vector space, $(Y, \langle \cdot, \cdot \rangle)$ a real inner product space, $I : X \to \mathbf{R}$ a bounded above convex function and $\Phi : X \to Y$ an affine operator such that*

$$\inf_{z \in \Phi(X)} \langle z, y \rangle = -\infty$$

for all $y \in \Phi(X) \setminus \{0\}$.

Then, the set of all global minima of the function $x \to I(x) + \|\Phi(x)\|^2$ is contained in $\Phi^{-1}(0)$.

THEOREM 1.5. - *Let $(T, \mathcal{F}, \mu)$ be a non-atomic measure space, with $0 < \mu(T) < +\infty$, E a real Banach space, $f, g : E \to \mathbf{R}$ two functions, with f lower semicontinuous and g continuous, satisfying*

$$\sup_E f < +\infty \ , \inf_E g = -\infty \ , \sup_E g = +\infty$$

and

$$\max \left\{ \sup_{x \in E} \frac{|f(x)|}{1 + \|x\|^p}, \sup_{x \in E} \frac{|g(x)|}{1 + \|x\|^p} \right\} < +\infty$$

for some $p \geq 1$. Finally, assume that f has at most one global minimum and that no zero of g is a global minimum of f. For each $u \in L^p(T, E)$, set

$$J(u) = \int_T f(u(t)) d\mu + \left(\int_T g(u(t)) d\mu \right)^2 .$$

Then, the restriction of the functional J to any decomposable subset of $L^p(T, E)$ containing the constant functions has no global minima.

THEOREM 1.6. - *Let X be a convex subset of a real vector space, $I : X \to \mathbf{R}$ a convex function and $P : X \to]0, +\infty[$ a concave function.*

Then, for each $\mu > 0$, a point $u \in X$ is a global minimum of the function $x \to I(x) - \mu \log P(x)$ if and only if one has

$$I(u) \leq I(x) - \mu \left(\frac{P(x)}{P(u)} - 1 \right)$$

for all $x \in X$.

2. Notations

To state our results in a more compact form, we now fix some notations.

Here and in the sequel, X is a non-empty set, Λ, Y are two topological spaces, y_0 is a point in Y.

A family $\mathcal{N}$ of non-empty subsets of X is said to be a filtering cover of X if $\cup_{A \in \mathcal{N}} A = X$ and for each $A_1, A_2 \in \mathcal{N}$ there is $A_3 \in \mathcal{N}$ such that $A_1 \cup A_2 \subseteq A_3$.

We denote by $\mathcal{G}$ the family of all lower semicontinuous functions $\varphi : Y \to [0, +\infty[$, with $\varphi^{-1}(0) = \{y_0\}$, such that, for each neighbourhood V of y_0, one has

$$\inf_{Y \setminus V} \varphi > 0 . \tag{2}$$

Moreover, we denote by $\mathcal{H}$ the family of all functions $\Psi : X \times \Lambda \to Y$ such that, for each $x \in X$, $\Psi(x, \cdot)$ is continuous, injective, open, takes the value y_0 at a point λ_x and the function $x \to \lambda_x$ is not constant. Furthermore, we denote by $\mathcal{M}$ the family of all functions $J : X \to \mathbf{R}$ whose set of all global minima (noted by M_J) is non-empty.

Finally, for each $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$, we put

$$\theta(\varphi, \Psi, J) = \inf \left\{ \frac{J(x) - J(u)}{\varphi(\Psi(x, \lambda_u))} : (u, x) \in M_J \times X \text{ with } \lambda_x \neq \lambda_u \right\} .$$

3. Results and proofs

With such notations, our abstract criterion reads as follows:

THEOREM 3.1. - *Let $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$.*

Then, for each $\mu > \theta(\varphi, \Psi, J)$ and each filtering cover $\mathcal{N}$ of X, there exists $A \in \mathcal{N}$ such that

$$\sup_{\lambda \in \Lambda} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) < \inf_{x \in A} \sup_{z \in A} (J(x) - \mu\varphi(\Psi(x, \lambda_z))) .$$

PROOF. Let $\mu > \theta(\varphi, \Psi, J)$ and let $\mathcal{N}$ be a filtering cover of X. Choose $u \in M_J$ and $x_1 \in X$, with $\lambda_{x_1} \neq \lambda_u$, such that

$$J(x_1) - \mu\varphi(\Psi(x_1, \lambda_u)) < J(u) .$$

Due to the nature of $\mathcal{N}$, there exists $A \in \mathcal{N}$ such that $u, x_1 \in A$. We have

$$0 \leq \inf_{z \in A} \varphi(\Psi(x, \lambda_z)) \leq \varphi(\Psi(x, \lambda_x)) = 0$$

for all $x \in A$, and so, since u is a global minimum of J, it follows that

$$\inf_{x \in A} \sup_{z \in A} (J(x) - \mu\varphi(\Psi(x, \lambda_z))) = \inf_{x \in A} \left(J(x) - \mu \inf_{z \in A} \varphi(\Psi(x, \lambda_z)) \right)$$

$$= \inf_X J = J(u) \ . \tag{3}$$

Since the function $\varphi(\Psi(x_1, \cdot))$ is lower semicontinuous at λ_u, there are $\epsilon > 0$ and a neighbourhood U of λ_u such that

$$J(x_1) - \mu\varphi(\Psi(x_1, \lambda)) < J(u) - \epsilon$$

for all $\lambda \in U$. So, we have

$$\sup_{\lambda \in U} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) \leq \sup_{\lambda \in U} (J(x_1) - \mu\varphi(\Psi(x_1, \lambda))) \leq J(u) - \epsilon \ . \tag{4}$$

Since $\Psi(u, \cdot)$ is open, the set $\Psi(u, U)$ is a neighbourhood of y_0. Hence, by (2), we have

$$\nu := \inf_{y \in Y \setminus \Psi(u, U)} \varphi(y) > 0 \ . \tag{5}$$

Moreover, since $\Psi(u, \cdot)$ is injective, if $\lambda \notin U$ then $\Psi(u, \lambda) \notin \Psi(u, U)$. So, from (4) and (5), it follows that

$$\sup_{\lambda \in \Lambda \setminus U} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda)) \leq J(u) - \mu \inf_{\lambda \in \Lambda \setminus U} \varphi(\Psi(u, \lambda)) \leq J(u) - \mu\nu \ . \tag{6}$$

Now, the conclusion comes directly from (3), (4), (5) and (6). $\triangle$

REMARK 3.1. - From the conclusion of Theorem 3.1 it clearly follows that, for any set $D \subseteq \Lambda$ with $\lambda_x \in D$ for all $x \in A$, one has

$$\sup_{\lambda \in D} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) < \inf_{x \in A} \sup_{\lambda \in D} (J(x) - \mu\varphi(\Psi(x, \lambda))) \ .$$

REMARK 3.2. - From the definition of $\theta(\varphi, \Psi, J)$, it clearly follows that $u \in M_J$ if and only if u is a global minimum of the function $x \to J(x) - \theta(\varphi, \Psi, J)\varphi(\Psi(x, \lambda_u))$. So, when $\theta(\varphi, \Psi, J) > 0$, from knowing that

$$J(u) \leq J(x)$$

for all $x \in X$, we automatically get

$$J(u) \leq J(x) - \theta(\varphi, \Psi, J)\varphi(\Psi(x, \lambda_u))$$

for all $x \in X$, which is a much better inequality since $\varphi(y) > 0$ for all $y \in Y \setminus \{y_0\}$.

REMARK 3.3 - It is likewise important to observe that if $\theta(\varphi, \Psi, J) > 0$, then the function $x \to \lambda_x$ is constant in M_J. As a consequence, if $\theta(\varphi, \Psi, J) > 0$ and the function $x \to \lambda_x$ is injective, then J has a unique global minimum. In particular, note that $x \to \lambda_x$ is injective when $\Psi(\cdot, \lambda)$ is injective for all $\lambda \in \Lambda$.

REMARK 3.4. - Remarks 3.2 and 3.3 show the interest in knowing when $\theta(\varphi, \Psi, J) > 0$. Theorem 3.1 can also be useful for this. Indeed, if for some $\mu > 0$, there is a filtering cover $\mathcal{N}$ of X such that

$$\sup_{\lambda \in \Lambda} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) \geq \inf_{x \in A} \sup_{z \in A} (J(x) - \mu\varphi(\Psi(x, \lambda_z)))$$

for all $A \in \mathcal{N}$, then $\theta(\varphi, \Psi, J) \geq \mu$.

As we said in the Introduction, we now extend Theorem C to the case where Λ is a convex subset of an arbitrary Hausdorff topological vector space. Recall that, when U is a topological space, a function $\psi : U \to \mathbf{R}$ is said to be inf-compact if, for each $r \in \mathbf{R}$, the set $\{x \in U : \psi(x) \leq r\}$ is compact.

THEOREM 3.2. - *Let X be a topological space, E a real Hausdorff topological vector space, $\Lambda \subseteq E$ a convex set, $f : X \times \Lambda \to \mathbf{R}$ a function which is lower semicontinuous and inf-compact in X, and upper semicontinuous and concave in Λ. Assume also that (1) holds.*
Then, there exists $\lambda^* \in \Lambda$ such that the function $f(\cdot, \lambda^*)$ has at least two global minima.

PROOF. Denote by $\mathcal{S}_\Lambda$ the family of all finite-dimensional convex subsets of Λ. Arguing by contradiction, assume that, for each $\lambda \in \Lambda$, the function $f(\cdot, \lambda)$ has a unique global minimum. Fix $S \in \mathcal{S}_\Lambda$. We claim that

$$\sup_S \inf_X f = \inf_X \sup_S f . \tag{7}$$

Let n be the dimension of S. Since E is Hausdorff, there exists an affine homeomorphism Φ between $\text{aff}(S)$, the affine hull of S, and $\mathbf{R}^n$. Thus, $Y := \Phi(S)$ is a convex subset of $\mathbf{R}^n$ of dimension n, and so its interior is non-empty. Now, put

$$\tilde{f}(x, y) = f(x, \Phi^{-1}(y))$$

for all $(x, y) \in X \times Y$. Of course, the function $\tilde{f}$ is lower semicontinuous and inf-compact in X, while is upper semicontinuous and concave in Y. Therefore, since $\tilde{f}(\cdot, y)$ has a unique global minimum for all $y \in Y$, Theorem C ensures that $\tilde{f}$ does not satisfy (1), that is

$$\sup_Y \inf_X \tilde{f} = \inf_X \sup_Y \tilde{f} . \tag{8}$$

Of course, one has

$$\sup_Y \inf_X \tilde{f} = \sup_S \inf_X f$$

and

$$\inf_X \sup_Y \tilde{f} = \inf_X \sup_S f \ .$$

So, (7) comes directly from (8). Now, since f satisfies (1), we can fix r so that

$$\sup_\Lambda \inf_X f < r < \inf_X \sup_\Lambda f \ . \tag{9}$$

For each $S \in \mathcal{S}_\Lambda$, put

$$C_S = \bigcap_{\lambda \in S} \{x \in X : f(x, \lambda) \leq r\} \ .$$

Clearly, C_S is closed. Note also that C_S is non-empty. Indeed, otherwise, we would have

$$r \leq \inf_X \sup_S f$$

and so, by (7),

$$r \leq \sup_S \inf_X f$$

against (9). Of course, if $S_1, ..., S_k \in \mathcal{S}_\Lambda$, then there is $\hat{S} \in \mathcal{S}_\Lambda$ such that

$$\bigcup_{i=1}^k S_i \subseteq \hat{S} \ .$$

So

$$C_{\hat{S}} \subseteq \bigcap_{i=1}^k C_{S_i} \ .$$

In other words, the family of closed compact sets $\{C_S\}_{S \in \mathcal{S}_\Lambda}$ has the finite intersection property, and so its intersection is non-empty. Hence, there is $\hat{x} \in X$ such that

$$f(\hat{x}, \lambda) \leq r$$

for all $\lambda \in S$ and all $S \in \mathcal{S}_\Lambda$. Consequently

$$\inf_X \sup_\Lambda f \leq r \ ,$$

against (9). Such a contradiction completes the proof. $\triangle$

A joint, direct application of Theorems 3.1 and 3.2 gives

THEOREM 3.3. - *Let $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$. Moreover, assume that X is a topological space, that Λ is a real Hausdorff topological vector space and that $\varphi(\Psi(x, \cdot))$ is convex for each $x \in X$. Finally, let $\mu > \theta(\varphi, \Psi, J)$ and $\mathcal{N}$ be a filtering cover of X such that, for each $A \in \mathcal{N}$, the function $x \to J(x) - \mu\varphi(\Psi(x, \lambda))$ is lower semicontinuous and inf-compact in A for all $\lambda \in \mathrm{conv}(\{\lambda_x : x \in X\})$.*

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Under such hypotheses, there exist $A \in \mathcal{N}$ and $\lambda^* \in \mathrm{conv}(\{\lambda_x : x \in A\})$ such that the restriction of the function $x \to J(x) - \mu\varphi(\Psi(x, \lambda^*))$ to A has at least two global minima.

PROOF. For each $(x, \lambda) \in X \times \Lambda$, put

$$f(x, \lambda) = J(x) - \mu\varphi(\Psi(x, \lambda)) \; .$$

Recalling that $\Psi(x, \cdot)$ is continuous and φ is lower semicontinuous, we have that, for each $x \in X$, the function $f(x, \cdot)$ is upper semicontinuous and concave in Λ. By Theorem 3.1, there exists $A \in \mathcal{N}$ such that

$$\sup_{\lambda \in D} \inf_{x \in A} f(x, \lambda) < \inf_{x \in A} \sup_{\lambda \in D} f(x, \lambda) \; ,$$

where

$$D = \mathrm{conv}(\{\lambda_x : x \in A\}) \; .$$

Now, the conclusion comes directly applying Theorem 3.2 to the restriction of f to $A \times D$. $\triangle$

On the basis of Theorem 3.3, we can give the proofs of Theorems 1.1-1.3.

Proof of Theorem 1.1. Arguing by contradiction, assume that X contains at least two points. Now, apply Theorem 3.3 taking: $\Lambda = Y$, $y_0 = 0$, $\varphi(x) = \|x\|$, $\Psi(x, \lambda) = x - \lambda$, $J = 0$ and $\mathcal{N} = \{X\}$. Note that we are allowed to apply Theorem 3.3 since $x \to \lambda_x$ is not constant. Then, it would exists $\lambda^* \in \mathrm{conv}(X)$ such that the function $x \to -\|x - \lambda^*\|$ has at least two global minima in X, against the hypotheses. $\triangle$

REMARK 3.5. - Observe that Theorem 1.1 improves a classical result by V. L. Klee ([1]) under two aspects: Y does not need to be complete and $\overline{\mathrm{conv}}(X)$ is replaced by $\mathrm{conv}(X)$. Note also that our proof is completely different from that of Klee which is based on the Schauder fixed point theorem.

Proof of Theorem 1.2. Let $\mu \in]2\theta, 2\eta[$. We clearly have

$$\lim_{\|x\| \to +\infty} \left(J(x) - \frac{\mu}{2}\|x - \lambda\|^2 \right) = +\infty \tag{10}$$

for all $\lambda \in X$. So, since X is finite-dimensional, the function $x \to J(x) - \frac{\mu}{2}\|x - \lambda\|^2$ is continuous and inf-compact for all $\lambda \in X$. Therefore, we can apply Theorem 3.3 taking: $X = Y = \Lambda$, $y_0 = 0$, $\varphi(y) = \|y\|^2$, $\Psi(x, \lambda) = x - \lambda$ and $\mathcal{N} = \{X\}$. Consequently, there exists $\lambda_\mu^* \in X$, such that the function $x \to J(x) - \frac{\mu}{2}\|x - \lambda_\mu^*\|^2$ has at least two global minima. By (10) and the finite-dimensionality of X again, the same function satisfies the Palais-Smale condition, and so it admits at least three critical points, thanks to Corollary 1 of [4]. Of course, this gives the conclusion, taking $y_\mu = \lambda_\mu^*$. $\triangle$

REMARK 3.6. - Clearly, there are two situations in which Theorem 1.2 can immediately be applied: when $\eta = +\infty$, and when $\eta > 0$ and $\theta = 0$. Note that one has $\theta = 0$ if, in particular, J possesses at least two global minima.

REMARK 3.7. - It is also clear that under the same assumptions as those of Theorem 1.2 but the finite-dimensionality of X, the conclusion is still true for every $\mu \in]2\theta, 2\eta[$ such that, for each $\lambda \in X$, the functional $x \to J(x) - \frac{\mu}{2}\|x - \lambda\|^2$ is weakly lower semicontinuous and satisfies the Palais-Smale condition.

Proof of Theorem 1.3. First of all, observe that $\varphi \in \mathcal{G}$, with $y_0 = 0$. Indeed, let $V \subset Y$ be a neighbourhood of 0 and let $s > 0$ be such that $B(0, s) \subseteq V$. Set

$$\alpha = \inf_{\|x\|=s} \varphi(x) \ .$$

Since $\dim(Y) < \infty$, $\partial B(0, s)$ is compact and so $\alpha > 0$. Let $x \in Y$ with $\|x\| > s$. Let S be the segment joining 0 and x. By convexity, we have

$$\varphi(z) \leq \varphi(x)$$

for all $z \in S$. Since S meets $\partial B(0, s)$, we infer that $\alpha \leq \varphi(x)$. Hence, we have

$$\alpha \leq \inf_{\|x\|>s} \varphi(x) \leq \inf_{x \in X \setminus V} \varphi(x) \ .$$

Now, fix $x_0 \in Y$ with $J'(x_0) \neq 0$. Taking into account that $\varphi'(0) = 0$, by continuity, we can choose $\sigma > 0$ so that

$$\|\varphi'(\lambda)\| < \|J'(x)\|$$

for all $(x, \lambda) \in B(x_0, \sigma) \times B(0, \sigma)$. For each $(x, \lambda) \in Y \times Y$, put

$$f(x, \lambda) = J(x) - \varphi(x - x_0 - \lambda) \ .$$

Of course, we have

$$f'_x(x, \lambda) \neq 0$$

for all $(x, \lambda) \in B\left(x_0, \frac{\sigma}{2}\right) \times B\left(0, \frac{\sigma}{2}\right)$. Next, since J' is locally Lipschitzian at x_0 and φ' is locally Lipschitzian at 0, there are $\rho \in \left]0, \frac{\sigma}{2}\right]$ and $L > 0$ such that

$$\|f'_x(x, \lambda) - f'_x(y, \lambda)\| \leq L\|x - y\|$$

for all $x, y \in B(x_0, \rho)$, $\lambda \in B(0, \rho)$. Now, fix $\lambda \in B(0, \rho)$. Denote by Γ_λ the set of all global minima of the restriction of the function $x \to f(x, \lambda) + \frac{L}{2}\|x - x_0\|^2$ to $B(x_0, \rho)$. Note that $x_0 \notin \Gamma_\lambda$ (since $f'_x(x_0, \lambda) \neq 0$). As f is continuous, the multifunction $\lambda \to \Gamma_\lambda$ is upper semicontinuous and so the function $\lambda \to \operatorname{dist}(x_0, \Gamma_\lambda)$ is lower semicontinuous. As a consequence, by compactness, we have

$$\delta := \inf_{\lambda \in B(0,\rho)} \operatorname{dist}(x_0, \Gamma_\lambda) > 0 \ .$$

At this point, from the proof of Theorem 1 of [6] it follows that, for each $\lambda \in B(0, \rho)$ and each $r \in]0, \delta[$, the restriction of function $f(\cdot, \lambda)$ to $B(x_0, r)$ has a unique global minimum. Fix $r \in]0, \delta[$. Apply Theorem 3.3 with $X = B(x_0, r)$, $\Lambda = Y$, $\mathcal{N} = \{B(x_0, r)\}$ and

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$\Psi(x, \lambda) = x - x_0 - \lambda$. With such choices, its conclusion does not hold with $\mu = 1$ (recall, in particular, that $r < \rho$). This implies that $1 \leq \theta(\varphi, \Psi, J)$ since the other assumptions are satisfied. But the above inequality is just equivalent to

$$J(u_r) \leq J(x) - \varphi(x - u_r)$$

for all $x \in B(x_0, r)$, where u_r is the unique global minimum of $J_{|B(x_0,r)}$, and the proof is complete. $\triangle$

Before proving Theorem 1.4, we point out another consequence of Theorem 3.1.

THEOREM 3.4. - *Let Y be a inner product space, and let $I : X \to \mathbf{R}$, $\Phi : X \to Y$ and $\mu > 0$ be such that the function $x \to I(x) + \mu\|\Phi(x)\|^2$ has a global minimum.*
Then, at least one of the following assertions holds:
(a) for each filtering cover $\mathcal{N}$ of X, there exists $A \in \mathcal{N}$ such that

$$\sup_{\lambda \in Y} \inf_{x \in A} (I(x) + \mu(2\langle \Phi(x), \lambda \rangle - \|\lambda\|^2)) < \inf_{x \in A} \sup_{\lambda \in \Phi(A)} (I(x) + \mu(2\langle \Phi(x), \lambda \rangle - \|\lambda\|^2)) \; ;$$

(b) for each global minimum u of $x \to I(x) + \mu\|\Phi(x)\|^2$, one has

$$I(u) \leq I(x) + 2\mu(\langle \Phi(x), \Phi(u) \rangle - \|\Phi(u)\|^2)$$

for all $x \in X$.

PROOF. Take $\Lambda = Y$, $y_0 = 0$. For each $x \in X$, $y, \lambda \in Y$, set

$$\varphi(y) = \|y\|^2 \; ,$$

$$\Psi(x, \lambda) = \Phi(x) - \lambda$$

and

$$J(x) = I(x) + \mu\|\Phi(x)\|^2 \; .$$

So that

$$J(x) - \mu\varphi(\Psi(x, \lambda)) = I(x) + \mu(2\langle \Phi(x), \lambda \rangle - \|\lambda\|^2) \; .$$

With these choices, (b) is equivalent to the inequality

$$\mu \leq \theta(\varphi, \Psi, J) \; .$$

Now, the conclusion is a direct consequence of Theorem 3.1. $\triangle$

A remarkable consequence of Theorem 3.4 is as follows:

THEOREM 3.5. - *Let Y be a inner product space, and let $I : X \to \mathbf{R}$, $\Phi : X \to Y$ be such that*
(i) $\sup_X I < +\infty$;
(ii) $\inf_{z \in \Phi(X)} \langle z, y \rangle = -\infty$ for all $y \in \Phi(X) \setminus \{0\}$.
Under such hypotheses, for each $\mu > 0$, at least one of the following assertions holds:

(a) *for each filtering cover $\mathcal{N}$ of X, there exists $A \in \mathcal{N}$ such that*

$$\sup_{\lambda \in Y} \inf_{x \in A} (I(x) + \mu(2\langle\Phi(x), \lambda\rangle - \|\lambda\|^2)) < \inf_{x \in A} \sup_{\lambda \in \Phi(A)} (I(x) + \mu(2\langle\Phi(x), \lambda\rangle - \|\lambda\|^2)) \; ;$$

(b) *the set of all global minima in X of the function $x \to I(x) + \mu\|\Phi(x)\|^2$ is contained in $\Phi^{-1}(0)$.*

PROOF. Assume that (a) does not hold. Then, we have to show that (b) holds. So, let $u \in X$ be a global minimum of the function $x \to I(x) + \mu\|\Phi(x)\|^2$. Then, by Theorem 3.4, we have

$$I(u) \le I(x) + 2\mu(\langle\Phi(x), \Phi(u)\rangle - \|\Phi(u)\|^2) \tag{11}$$

for all $x \in X$. Now, in view of (i) and (ii), if $\Phi(u) \ne 0$, we would have

$$\inf_{x \in X} (I(x) + 2\mu\langle\Phi(x), \Phi(u)\rangle) = -\infty$$

in contradiction with (11). Therefore, $\Phi(u) = 0$. $\qquad\qquad\qquad\qquad\qquad \triangle$

On the basis of Theorem 3.5, we can now give the

Proof of Theorem 1.4. First of all, note that the function $x \to I(x) + \langle\Phi(x), \lambda\rangle$ is convex in X for all $\lambda \in Y$. Now, set

$$\mathcal{N} = \{\text{conv}(B) : B \subset X, \ B \text{ finite}\} \; .$$

Clearly, $\mathcal{N}$ is a filtering cover of X. Fix $A \in \mathcal{N}$. Since Φ is affine, we have

$$\Phi(A) = \text{conv}(\Phi(B))$$

where B is a finite subset of X such that $\text{conv}(B) = A$. So, $\Phi(A)$ is a convex and compact subset of Y. Taking into account that, for each $x \in X$, the function $\lambda \to 2\langle\Phi(x), \lambda\rangle - \|\lambda\|^2$ is continuous and concave in Y, we can apply Kneser's minimax theorem ([2]). So, thanks to it, we have

$$\sup_{\lambda \in \Phi(A)} \inf_{x \in A} (I(x) + 2\langle\Phi(x), \lambda\rangle - \|\lambda\|^2) = \inf_{x \in A} \sup_{\lambda \in \Phi(A)} (I(x) + 2\langle\Phi(x), \lambda\rangle - \|\lambda\|^2) \; .$$

This shows that condition (a) of Theorem 3.5 does not hold. So, condition (b) of the same theorem holds, and the conclusion is reached. $\qquad\qquad\qquad\qquad\qquad \triangle$

REMARK 3.8. - Let $(Y, \langle\cdot, \cdot\rangle)$ be a real inner product space and let $\mathcal{R}$ denote the family of all subsets S of Y, containing more than one point, such that

$$\inf_{z \in S} \langle z, y\rangle = -\infty$$

for all $y \in S \setminus \{0\}$. In view of Theorems 3.5 and 1.4, it would be interesting to characterize the sets belonging to the family $\mathcal{R}$. One of such characterizations, for convex sets, is suggested by Theorem 1.4 itself. Actually, a convex set $S \subseteq Y$ belongs to $\mathcal{R}$ if and only

if, for each convex bounded above function $I : X \to \mathbf{R}$, the set of all global minima in S of the function $x \to I(x) + \|x\|^2$ is either $\emptyset$ or $\{0\}$. Concerning the proof, the "only if" part comes out directly from Theorem 1.4 taking $\Phi(x) = x$. For the "if" part, it is enough to observe that, if we take $y \in S \setminus \{0\}$, since y is the global minimum of the function $x \to -2\langle x, y\rangle + \|x\|^2$, by assumption, the convex function $x \to -2\langle x, y\rangle$ must be unbounded above in S, and so $S \in \mathcal{R}$. In [13], J. Saint Raymond proved that, when Y is a Hilbert space, a closed convex subset of Y (different from $\{0\}$) belongs to the family $\mathcal{R}$ if and only if it is a linear subspace. At the same time, he provided an example of a closed convex set in a non-complete space Y which belongs to $\mathcal{R}$ and does not contain 0. So, it would be of particular interest to find a geometric condition characterizing closed convex sets belonging to $\mathcal{R}$ which, when Y is complete, reduces to being a linear subspace.

The proof of Theorem 1.5 is based on Theorem 3.5 again.

Proof of Theorem 1.5. Let $X \subseteq L^p(T, E)$ be a decomposable set containing the constant functions. Arguing by contradiction, assume that $\hat{u} \in X$ is a global minimum of $J_{|X}$. For each $h \in L^p(T)$, set

$$X_h = \{u \in X : \|u(t)\| \le h(t) \text{ a.e. in } T\} .$$

Clearly, X_h is decomposable and the family $\{X_h\}_{h \in L^p(T)}$ is a filtering covering of X. For each $u \in X$, put

$$I(u) = \int_T f(u(t))d\mu$$

and

$$\Phi(u) = \int_T g(u(t))d\mu .$$

Note that our assumptions readily imply

$$\sup_X I < +\infty, \ \inf_X \Phi = -\infty, \ \sup_X \Phi = +\infty . \tag{12}$$

Fix $h \in L^p(T)$. Thanks to the Lyapunov convexity theorem, $\Phi(X_h)$ is an interval, and so

$$\Lambda_h := \overline{\Phi(X_h)}$$

is a compact interval. Therefore, taking into account that the function

$$(u, \lambda) \to I(u) + 2\lambda\Phi(u) - \lambda^2$$

is lower semicontinuous in $L^p(T, E)$, and continuous and concave in $\mathbf{R}$, we can apply Theorem 1.D of [7]. Thanks to it, we then have

$$\sup_{\lambda \in \Lambda_h} \inf_{u \in X_h} (I(u) + 2\lambda\Phi(u) - \lambda^2) = \inf_{u \in X_h} \sup_{\lambda \in \Lambda_h} (I(u) + 2\lambda\Phi(u) - \lambda^2) .$$

Hence, condition (a) of Theorem 3.5 does not hold, and so, in view of (12), condition (b) of it must hold. Hence, $\Phi(\hat{u}) = 0$, and so $\hat{u}$ is a global minimum of I. This implies that f

has a unique minimum, say ξ_0, and that $u(t) = \xi_0$ a.e. in T. Hence, $g(\xi_0) = 0$, against the assumptions. $\triangle$

A further consequence of Theorem 3.1 is the following:

THEOREM 3.6. - *Let $I, \Phi : X \to \mathbf{R}$ and $\mu > 0$ be such that the function $I - \mu\Phi$ has a global minimum.*

Then, at least one of the following assertions holds:
(a) for each filtering cover $\mathcal{N}$ of X, there exists $A \in \mathcal{N}$ such that

$$\sup_{\lambda \in \mathbf{R}} \inf_{x \in A} (I(x) - \mu(e^{\Phi(x)-\lambda} + \lambda)) < \inf_{x \in A} \sup_{\lambda \in \Phi(A)} (I(x) - \mu(e^{\Phi(x)-\lambda} + \lambda)) \ ;$$

(b) for each global minimum u of $I - \mu\Phi$, one has

$$I(u) \leq I(x) - \mu(e^{\Phi(x)-\Phi(u)} - 1)$$

for all $x \in X$.

PROOF. Take $\Lambda = Y = \mathbf{R}$, $y_0 = 0$. For each $x \in X$, $y, \lambda \in Y$, set

$$\varphi(y) = e^y - y - 1 \ ,$$

$$\Psi(x, \lambda) = \Phi(x) - \lambda$$

and

$$J(x) = I(x) - \mu\Phi(x) \ .$$

So that

$$J(x) - \mu\varphi(\Psi(x, \lambda)) = I(x) - \mu(e^{\Phi(x)-\lambda} + \lambda - 1) \ .$$

With these choices, (b) is equivalent to the inequality

$$\mu \leq \theta(\varphi, \Psi, J) \ .$$

Now, the conclusion is a direct consequence of Theorem 3.1. $\triangle$

Finally, we can give the

Proof of Theorem 1.6. The proof is very similar to that of Theorem 1.4. So, let $\mathcal{N}$ be as in that proof. Fix $A \in \mathcal{N}$. Since P is concave and positive, $P(A)$ is a bounded subset of $\mathbf{R}$ with positive infimum. For each $x \in X$, put

$$\Phi(x) = \log P(x) \ .$$

Now, fix a compact real interval $[a, b]$ such that $\Phi(A) \subseteq [a, b]$. Clearly, the function $(x, \lambda) \to I(x) - \mu(e^{\Phi(x)-\lambda} + \lambda)$ is convex in X, and concave and continuous in $\mathbf{R}$. So, by Kneser's minimax theorem again, we have

$$\sup_{\lambda \in [a,b]} \inf_{x \in A} (I(x) - \mu(e^{-\lambda}P(x) + \lambda)) = \inf_{x \in A} \sup_{\lambda \in [a,b]} (I(x) - \mu(e^{-\lambda}P(x) + \lambda)) \ .$$

This shows that condition (a) of Theorem 3.6 does not hold. So, condition (b) of the same theorem holds, and the conclusion is reached. $\triangle$

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